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Theorems on the Simple Finite Polygon and Polyhedron.[*]

By N. J. Lennes.

INTRODUCTION.

In his "Foundations of Geometry,"[†] Professor Hilbert defines the measure of area of a polygon as the sum of the measures of area of all the triangles into which a polygon may be decomposed by a definite decomposition. This definition implies:

1. The existence of a criterion as to what constitutes a decomposition of a polygon into triangles.
2. That for every polygon there exists a finite decomposition; that is, a decomposition resulting in a finite number of triangles.
3. That any two decompositions into triangles of the same polygon result in sets of triangles such that the sum of the measures of area of the triangles is the same for both sets.

A similar definition of the measure of volume of a polyhedron requires similar propositions about it.

The proofs of the theorems here implied require as lemmas the theorems that the polygon and polyhedron separate the plane and the three-space respectively into two mutually exclusive sets, and these in turn are based upon the fundamental theorem that a straight line divides a plane into two such sets.

Hilbert proves[‡] that the measure of area of a polygon as above defined is independent of any particular decomposition of the polygon. The corresponding theorem for the measure of volume of the polyhedron is proved by S. O. Schatunovsky.[§] That a simple polygon separates the remaining points in which it lies into two mutually exclusive sets has been proved by O. Veblen* and also by Hans Hahn.† But so far as known to me there is in the literature no proof that any polygon may be decomposed into a finite number of triangles.

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[*] Read before the Chicago Section of the American Mathematical Society at its April meeting, 1903. Some minor changes and additions have been made since that time.

[†] "Foundations of Geometry," by Professor David Hilbert, translated by Professor E. J. Townsend (Open Court Pub. Co., Chicago, Ill.).

[‡] *Loc. cit.*, pp. 57–66.

[§] *Mathematische Annalen*, Vol. LVII (1903), p. 496.

The fundamental theorem on the polyhedron — viz., that it separates space into two sets — has not been proved, nor has the theorem concerning its decomposability into tetrahedrons. Indeed it does not appear that a careful definition of a polyhedron has been formulated.

The object of this paper is to formulate such a definition and to give proofs of the theorems just enumerated. Two new proofs are given of the proposition that a simple polygon separates the plane into two sets. The corresponding theorem on the polyhedron may be proved in a manner analogous to either of these, but only one of these proofs is here carried through.

The polygon is decomposed into triangles in such manner that no new vertices are created; that is, every vertex of the resulting triangles is a vertex of the original polygon. It is shown, however, that in general a polyhedron can not be decomposed into tetrahedrons without creating new vertices.

In proving these theorems no use has been made of continuity, congruence or the axiom on parallels.

Axioms I–IX of Professor Veblen‡ or the axioms of Professor Hilbert, excluding those just mentioned, are sufficient, as are the Projective Axioms of Geometry given by Professor Moore.§

PART I. POLYGONS.

§ 1. *Preliminary Propositions.*

In this section are given a number of simple propositions that are required in the argumentation that follows. Many of these theorems are proved by Veblen.‖ Each of the others follows by simple argumentation from those that precede it. A number of these preliminary propositions are not needed before we reach the polyhedron.

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* *Transactions of the American Mathematical Society*, Vol. V (1904), pp. 343-380.

† *Monatshefte für Mathematik und Physik*, Vol. XIX, pp. 289–303.

‡ *Loc. cit.*

§ *Transactions of the American Mathematical Society*, Vol. III (1903), p. 142.

‖ *Loc. cit.*

1. *For any set of n distinct points on a line a notation* $A_1, A_2, \ldots, A_n$ *may be so arranged that the points* A_i, A_j, A_k *are in the order* $A_i\, A_j\, A_k$ *for all integral values of* i, j, k *such that* $i < j < k$ *and* $i \geqq 1$, $k \leqq n$.

DEFINITION. The points on a line between two of its points A, B constitute the *segment* AB. The points A and B are the *end-points* of the segment. The segment is said to *connect* its end-points.

2. *A point separates the remaining points of a line on which it lies into two sets of points such that a segment connecting points of the same set does not contain the point, while a segment connecting points of different sets does contain the point.*

3. *Any line of a plane separates the remaining points of the plane into two sets such that a segment connecting two points of the same set contains no point of the line, while a segment connecting points of different sets contains a point of the line.*

DEFINITIONS. The points of a line which lie on the same side of a point A of the line are called a *half-line* or *ray*. The ray is said to *proceed* from A. If B is any point of the ray, then the ray is referred to as the ray AB (not BA). A is the *end-point* of the ray. The two parts into which a line separates the remaining points of a plane are called *half-planes*.

A ray lying in the same line as the ray AB and proceeding from A but not containing B is the *complementary* ray of AB. Two non-complementary rays proceeding from the same point A together with the point A form an *angle* whose *vertex* is A. The two rays together with the vertex are sometimes referred to as the *boundary* of the angle. If the two rays are denoted by h and k, the angle formed by them is often denoted by $\angle\,(h, k)$.

4. *An angle separates the remaining points of the plane in which it lies into two sets, an interior and an exterior set, such that a segment connecting an interior and an exterior point contains one point of the angle, a segment connecting two interior points contains no point of the angle, while a segment connecting two exterior points and not containing the vertex contains two points or no point of the angle.*

5. *A segment connecting points one on each side of an angle lies entirely within the angle.*

6. *A ray proceeding from the vertex of an angle and containing a point within the angle lies entirely within it. If a segment connects points on different sides of the angle, it contains a point of this ray.*

7. *A ray proceeding from the vertex of an angle and lying within it forms with the sides of the angle two angles which have no interior point in common. Every point within the first angle lies within one of the new angles or lies on their*

common side, and every point within either of the new angles lies within the original angle.

8. *If a finite number of rays proceed from the vertex of an angle and lie within it, they may be ordered so as to form a series of angles no two of which have an interior point in common.*

PROOF. Let a segment connect points on different sides of the original angle. By (6) this segment meets each ray. Order the intersection points by (1) and apply (7).

DEFINITIONS. If the points A_1, A_2, A_3 are not collinear, the segments A_1A_2, A_2A_3, A_3A_1, together with the points A_1, A_2, A_3, form a *triangle*. The segments are its *sides* and the points A_1, A_2, A_3 are its *vertices*. The points on the set of all segments whose extremities lie on different sides of a triangle constitute the *interior* points of the triangle. Points of the plane not on or within the triangle constitute its *exterior* points.

9. *A triangle so separates its interior and exterior points that a segment connecting an interior and an exterior point of the triangle meets it in one point, a segment connecting two interior points does not meet the triangle, while a segment connecting two exterior points and not containing a vertex meets the triangle in two points or in no point.*

Any number of rays proceeding from the same point P may be ordered by constructing a triangle of which P is an interior point and noting the order of the points on the triangle in which the rays meet it.

10. *The order of a set of distinct rays proceeding from a point is independent of the triangle used in ordering them.*

PROOF. Let a, b, c, d be half-lines proceeding from a point P. Suppose a does not make a straight line with any other of the half-lines. Then at least two of them, as b and c, lie on the same side of the line determined by a. Then one of the three half-lines a, b, c lies within the angle formed by the other two. Hence in this case the order is definite. If a and c form a straight line, then if b and d are on the same side of this line the proposition follows as before. If they are on opposite sides it likewise follows.

11. *A segment containing a point within an angle (or triangle) meets it in two points, or else at least one end-point of the segment lies on or within the angle (triangle).*

12. *If two triangles have an interior point in common, then one of the triangles is wholly or partly within the other.*

13. *A segment connecting a vertex of a triangle with a point in its opposite side*

lies within the triangle and forms with it two triangles which have no interior points in common. The interior points of the new triangles, together with their common side, are identical with the interior points of the original triangle.

14. *The interior points of a triangle, and no other points, lie within each of the angles of the triangle.*

15. *If* $[P]$* *is a finite set of points within an angle* BAC, *and if* B' *is any point on the ray* AB, *then there is a point* C' *on the ray* AC *such that there is no point of* $[P]$ *on or within the triangle* $AB'C'$.

PROOF. Draw segments from B' through each of the points of $[P]$ which are within the triangle $AB'C$ meeting AC in points P_i'. Then among the rays $B'P_i'$ there is one ray $B'P_1'$ such that no ray $B'P_i'$, and hence no point of $[P]$, lies within the angle $AB'P_1'$ (8). Let C' be any point of AP_1'. Then C' is a point such as is required (14), (13).

16. *If* $[\sigma]$ *is any finite set of segments and* ABC *a triangle which has no point in common with* $[\sigma]$ *or its end-points, except possibly on* BC, *and if there are points of* $[\sigma]$ *within* ABC, *then there is an end-point* P *of a segment of* $[\sigma]$ *within* ABC *such that there is no point of* $[\sigma]$ *on or within the triangle* ABP, *and no end-point of* $[\sigma]$ *on it except the point* P.

PROOF. By (11) there are end-points $[P]$ of segments of $[\sigma]$ within the triangle ABC. As in the proof of (15) find a point P_1 of $[P]$ such that there is no point of $[P]$ within the angle ABP_1. If P_1 is the only point of $[P]$ on the ray BP_1, then P_1 is the required point P. If not, let P_2 be that point of $[P]$ on the ray BP_1 which is nearest B. Then this is the required point P (14), (11).

DEFINITION. A point P is an *interior* point of a set of points $[P]$ if there exists a triangle t containing P as an interior point such that every other interior point of t is a point of $[P]$.

If every point of a set $[P]$ is an interior point of the set, then $[P]$ is said to be an *entirely open set.*

17. *If* $[P]$ *is a set of points consisting of a finite set of segments* $[\sigma]$ *together with their end-points and any other finite set of points* $[P]'$, *then the remaining points of the plane constitute an entirely open set.*

PROOF. Let Q be any point not in $[P]$ and let l be a line through Q not containing an end-point of a segment of $[\sigma]$. Then l contains only a finite

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* The notation $[P]$ is used to denote a set or class an element of which may be denoted by the symbol within the brackets.

number of points of $[P]$. Hence by (1) there is a segment AD on l containing Q but no point of $[P]$, A and D not being points of $[P]$. Through D pass a line l_1, distinct from l and not containing an end-point of $[\sigma]$. Then there is a segment $B'C'$ on l_1 containing D but no point of $[P]$. Then by (15) there are points B and C on $B'D$ and DC', respectively, such that no point of $[P]'$ and no end-point of a segment of $[\sigma]$ lies on or within either of the triangles ABD and ACD, and therefore neither on nor within the triangle ABC (13). Hence, by (11), there is no point of $[\sigma]$ on or within the triangle ABC, and therefore no point of $[P]$.

§ 2. *Separation of the Plane by a Polygon.*

DEFINITIONS. The segments A_1A_2, A_2A_3, ..., $A_{n-1}A_n$, together with the points $A_1, A_2, \ldots, A_n$, form a *broken line* A_1A_n. The segments are the *sides* of the broken line and the points $A_1, A_2, \ldots, A_n$ are its *vertices.* If no two sides have a point in common, no vertex lies on a side, and no two vertices coincide, except possibly A_1 and A_n, the broken line is said to be *simple.* If A_1 and A_n coincide, and if A_1A_n is a simple broken line, it is said to form a *simple polygon.*

If for any two points P_1 and P_2 of an entirely open set $[P]$ there exists a broken line P_1P_2 lying in $[P]$, then $[P]$ is said to be *an open connected region.*

A point R is said to be *finitely accessible,* or simply *accessible,* from a point Q with respect to a set of points $[P]$, if there exists a broken line RQ containing no point of $[P]$, except possibly R or Q, or both. R and Q are also said to be *mutually accessible* with respect to $[P]$.

18. *An angle and a polygon have an even number of points in common provided that no vertex of either lies on the other.*

PROOF. In the polygon A_1A_2, A_2A_3, ..., A_nA_1 let A_1 be an exterior point of the angle (h, k). Following the sides of the polygon in the order given let A_i be the first vertex from A_1 which lies within the angle (h, k). Then the broken line A_1A_i meets the angle once (4). In this manner we can show that if A_j is the next vertex after A_i which lies outside the angle, then A_iA_j meets the angle once, etc. Since the number of sides of the polygon is finite, this process leads to the starting-point A_1 after meeting the angle an even number of times. (Zero is here regarded as an even number.)

DEFINITION. A point P not on a polygon is said to be an interior point of the polygon if a ray proceeding from P and not containing a vertex of the polygon meets it in an odd number of points. The point P is an exterior point of the polygon if such ray meets it in an even number of points.

That the external or internal character of the point as here defined depends *upon the point and the polygon only,* and not upon what particular ray is used, follows from (18). For if h and k are any rays proceeding from the same point and not containing vertices, it follows that h meets the polygon an odd or an even number of times according as k does.

If two points are both interior or both exterior they are said to lie on the *same side* of the polygon.

19. *If a broken line contains no point of a polygon, then all its points lie on the same side of the polygon.*

PROOF. Consider a broken line AB, BC, ..., KL which does not meet a certain polygon p. If either of the rays AB or BA contains no vertex of p, then by the definition of interior and exterior points all points of the segment AB, together with its end-points, lie on the same side of p. If each of these rays contains a vertex of p, proceed as follows:

From A', any point in the segment AB or the end-point A, construct a ray $A'K$ containing no vertex of p and hence not in the line AB. Then, by (1), (15) and (11), there is a point C on $A'K$, and not on p, such that there is no point of p on $A'C$ and BC, and such that the ray BC contains no vertex of p. Then B, C and A' are on the same side of p. Since A' is any point of AB, including A, it follows that every point of the segment AB, including the end-point A, is on the same side of p as the point B.

Continuing in this way, we prove that the whole broken line lies on the same side of p.

20. *A broken line connecting an interior and an exterior point of a polygon meets it at least once.*

21. *If a broken line connects two exterior or two interior points of a polygon, and if neither contains a vertex of the other, then they have an even number of points in common.*

22. *If a segment AB meets a polygon in only one point C which is not a vertex, then the segments AC and CB lie on opposite sides of the polygon.*

23. *A polygon has an exterior and also an interior set of points.*

24. *Every point of the plane not on a polygon is either internal or external to it.*

25. *If two points are on the same side of a polygon, they are mutually accessible with respect to the polygon.*

PROOF. Consider two exterior points A and B of a polygon p. Connect these with points C and D, respectively, on p, it being understood that there are no points of p on the segments AC and BD. About each vertex A_i of p construct a triangle t_i having the following properties: On or within each triangle there is no point of another of the triangles or of the polygon p except points on the segments $A_{i-1} A_i$, $A_i A_{i+1}$ and the point A_i (17). The segments AC and BD also lie entirely outside each triangle. No triangle t_i has a vertex on the polygon.

Suppose that C lies on $A_1 A_2$ of the polygon. Let Q be a point on $A_1 A_2$ within t_2. Then, by (15), (11), there is a point R on AC such that RQ contains no point of p. Let RQ meet t_2 in S_2. Then A and S_2 are mutually accessible. In precisely the same manner we show there are points on t_2 and t_3, both exterior to p, which are mutually accessible. Denote these by S_2' and S_3 respectively. But S_2' and S_2 are mutually accessible with respect to p; for, tracing the triangle t_2 from S_2 to S_2', we meet p in two points or in no point (S_2 and S_2' both being exterior), and hence one way along t_2 from S_2 to S_2' fails to meet p. Hence A and S_3 are mutually accessible. Proceeding in this way we finally show that A and B are mutually accessible. In case A and B are both interior, the argument is precisely the same.

As a summary of (23), (24), (20), (25), (17) we now have:

26. *A simple polygon separates the remaining points of the plane in which it lies into two entirely open connected sets such that every broken line connecting points not in the same set meets the polygon.*

Theorem (26) has a certain form of converse as follows:

27. *If p consists of a finite set of segments* $[\sigma]$ *together with their end-points, and if p separates the remaining points of the plane into two sets while no proper*[*] *subset of p does thus separate it, then p is a simple polygon.*

PROOF. (a) *Every end-point of a segment of p must be common to at least two segments.* For suppose in the segment $A_1 A_2$ the point A_1 is an end-point of this segment only. Then if P and Q are not mutually accessible with respect to p, they are not mutually accessible with respect to a set consisting of p with the segment $A_1 A_2$ removed; for if there exists a broken line connecting P and Q which contains no point of p except points of $A_1 A_2$, then by suitable use of the process employed in the proof of (25) we can find a broken line PQ which goes *around* the end-point A_1, and which therefore contains no point of p.

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[*] A set is a proper subset of another set if it consists of some, but not all, the elements of that set.

(b) It follows that in p we may trace a broken line $A_1 A_2$, $A_2 A_3$, ... without ever coming to a vertex from which we can not proceed without retracing the segment last traced. Since the set of segments in p is finite, we must finally reach a point in the broken line already traced. But this gives a simple polygon. Since this polygon separates the plane, it follows that it contains every point of p; for by hypothesis no proper subset of p separates the plane.

§ 3. *Decomposition of the Polygon.*

DEFINITION. A vertex A_k of a polygon is said to be *projecting* if there exists a line which contains a point of each of the sides $A_{k-1} A_k$ and $A_k A_{k+1}$ and no other point of the polygon.

28. *Every polygon has at least one projecting vertex.*

PROOF. Let p be the given polygon and $[\sigma]$ the set of all segments whose end-points are vertices of p. Let l be a line meeting sides of the polygon but containing none of its vertices. Let P be a point on l collinear with no two vertices of p such that all intersection points of l with segments of $[\sigma]$ are on the same side of P (1). From P construct rays through each of the vertices of p. By means of (5) and (8) it follows that there are two of these rays, h and k, one on each side of l, each of which forms an angle with l within which lies no vertex of p and hence no point of p (11). Let h and k meet p in the vertices A_h and A_k respectively. Then A_h and A_k are projecting vertices; for, let h' be the ray next to h. Then any ray proceeding from P and lying within the angle (h', h) meets p in the sides $A_{h-1} A_h$ and $A_h A_{h+1}$ and in no other points (6), (11).

29. *For any polygon there is a line such that the polygon lies entirely on the same side of it.*

PROOF. In the construction used in the preceding proof extend the segment $A_h A_k$ to a point Q and draw the line PQ. Now the ray PQ can not meet p, since Q is outside the angle (h, k). Moreover, the point Q and the polygon p lie on the same side of the line PA_h (with the exception of the point A_h, which lies on the line). Hence that part of the line PQ which lies on the side of the line PA_h opposite to Q can not meet p. That is, PQ is the required line.

DEFINITION. A set of triangles $[t]$ is said to constitute a decomposition of a polygon p if: (a) no two triangles of $[t]$ have an interior point in common; (b) every interior point of p lies on or within a triangle of $[t]$; (c) every interior point of a triangle of $[t]$ is an interior point of p.

30. PROBLEM. *To decompose a given polygon into a set of triangles* $[t]$ *such that every vertex of* $[t]$ *is a vertex of the given polygon.*

SOLUTION. Let p be the given polygon and A_i a projecting vertex. Two cases are possible, viz.: (a) There may be vertices of p within the triangle $A_{i-1} A_i A_{i+1}$ or on the side $A_{i-1} A_{i+1}$ of this triangle; (b) there may be no vertex of p within the triangle $A_{i-1} A_i A_{i+1}$ or on $A_{i-1} A_{i+1}$.

In case (a), there is by (16), or by (1), a vertex A_h of p within $A_{i-1} A_i A_{i+1}$, or on $A_{i-1} A_{i+1}$, such that no vertex of p except A_h lies on or within the triangle $A_i A_{i+1} A_h$. Then p and the segments $A_i A_h$ and $A_{i+1} A_h$ form the triangle $A_i A_{i+1} A_h$ and one or two polygons, depending on whether $A_{i+1} A_h$ is a side of p or not. Denote, by $[\sigma]$, p together with the segments $A_i A_h$ and $A_{i+1} A_h$. By repeating the above process we obtain another triangle $A_j A_{j+1} A_k$ such that no point of $[\sigma]$ lies within it, and so on.

In case (b), A_{i-1}, A_i, A_{i+1} form a triangle such that no point of p lies within it or on the side $A_{i-1} A_{i+1}$. Then this last segment, together with p, forms a triangle and a polygon. In either case if the result is a triangle and one polygon, one new side is added, and this is common to the triangle and the polygon. Hence the new polygon has one side less than the original. If the result is a triangle and two polygons, two new segments are added and each of these is common to the triangle and one new polygon. Hence the two resulting polygons have together one side more than the original polygon. In any case this process will result in $n-2$ triangles, n being the number of sides of the polygon. Denote this set by $[t]$. Then:

(a) No two triangles of $[t]$ have an interior point in common (12).

(b) Every side of p is a side of exactly one triangle of $[t]$.

(c) Every side of a triangle of $[t]$ which is not a side of p is a side of exactly two triangles of $[t]$.

(d) Any two complementary* subsets of $[t]$ are such that there is at least one triangle in each set which has a side in common with a triangle of the other set. (Such sides are not sides of p.) (b), (c) and (d) follow directly from the construction.

(e) Any broken line which does not meet p lies entirely on or within triangles of $[t]$ if a single point of it lies on or within such triangle. This is

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[*] Two sets are said to be complementary subsets of a given set if, while having no common elements, the two sets together consist of the same elements as the given set.

obvious; for tracing such broken line across a side of one triangle leads into another, since each side which this broken line can meet is a common side of two triangles having no interior point in common (12).

(f) Any two points within triangles of $[t]$ or on sides of such triangles are mutually accessible with respect to p. This is an immediate consequence of (d).*

(g) Every interior point of p lies on or within a triangle of $[t]$. For by construction the interior points of the triangle $A_i A_{i+1} A_h$ are interior points of p. Hence, by (e) and (26), (g) follows.

It follows therefore that $[t]$ is a decomposition of p.

§ 4. *Another Proof that a Polygon Separates a Plane.*

We now assume the propositions of § 1; propositions (28), (29) and (a)–(f) under (30) of § 3; but not those of § 2. We refer to a set of triangles obtained from a given polygon as in (30), having the properties (a)–(f) there enumerated, as the set $[t]$ of that polygon.

31. *Any two points of a polygon are mutually accessible with respect to the polygon by means of a broken line lying on or within triangles of* $[t]$.

PROOF. Connect the given points with points within triangles of $[t]$ and apply (30, f).

32. *A point within a triangle of* $[t]$ *is not accessible with respect to* p *from a point not on or within such triangle* (30, e).

33. *Any point in the plane and any point on a polygon are mutually accessible with respect to the polygon.*

PROOF. If the given point lies on the polygon or on or within a triangle of $[t]$, this is proposition (30, f). If the point is exterior to every triangle of $[t]$, proceed as follows: From the point P obtained in the proof of (28) draw rays to each vertex of p. Order these rays and on each of them select points $B_1, B_2, \ldots, B_n$ such that all intersection points of the ray PB_i and the set $[\sigma]$ of (28) lie beweeen P and B_i. Extend $A_h A_k$ of (28) to A' and A'', respectively. Then the broken line PA', $A'B_1$, $B_1 B_2, \ldots, B_{n-1}$, B_n, $B_n A''$, $A''P$ forms a simple polygon p_1 which does not meet p. Connect two points A and B on PA' with points C and D on the same segment of p in such wise that the segments AC and BD do not meet. Then the polygon p, omitting the segment CD, and

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* Note that in § 3, up to this point, no use has been made of § 2.

the polygon p_1 omitting the segment AB, plus the segments AC and BD, form a simple polygon p_2. From this polygon we obtain a set of triangles $[t]_2$. It may readily be shown that the segment CD is not on or within one of these triangles, but that it is accessible from P. Hence, by (32) and (30, f), the triangles $[t]$ of the original polygon and the triangles of $[t]_2$ have no interior point in common. By (31) P is accessible to every point of p_2 and hence to every point of p. We now show that any point Q not in or on a triangle of $[t]$ is accessible from P. Construct a segment QR with R on a side $A_i A_{i+1}$ of p but no point of p on QR. Also connect P with R by means of a broken line $PP_1, \ldots, P_n R$ containing no point of p. Now QR and $P_n R$ do not lie within a triangle of $[t]$ and hence are on the same side of $A_i A_{i+1}$. Using (15), connect these segments by means of a segment not meeting p. Then we have a broken line connecting P and Q which contains no point of p. But every point of p is accessible from P and hence from Q.

DEFINITION. A line which does not meet a polygon, together with all points accessible from it with respect to the polygon and not on the polygon, are *exterior points* of the polygon. All the remaining points of the plane not on the polygon are *interior points.*

34. *Any point in a plane not on a given polygon is exterior to it or lies on or within a triangle of* $[t]$.

PROOF. Connect the given point Q with a point on p by means of a segment QR. If QR does not lie within a triangle of $[t]$, then by (33), (29) it is accessible from an exterior line and hence an exterior point. If QR has points within a triangle of $[t]$, then by (30, e) Q is on or within such triangle.

From the propositions (31)–(34) and (30, f) we have:

35. *A simple polygon separates the remaining points of the plane into two entirely open connected sets.*

PART II. POLYHEDRONS.

§ 5. *Definition of Polyhedron and Preliminary Propositions.*

DEFINITION. A *simple polyhedron,* or simply *polyhedron,* is a set of points consisting of a finite set $[t]$ of triangles, together with their interior points, having the following properties:

(1) Every side of a triangle of $[t]$ is common to an even number of triangles of this set.

(2) There exists no finite decomposition of the triangles of $[t]$ into other triangles forming a set $[t]'$ such that (1) is true of a proper subset of $[t]'$.

In this definition (2) is to be so understood that $[t]'$ may be the same set as $[t]$.

The vertices of the triangles are the *vertices* of the polyhedron. The sides of the triangles are the *edges* of the polyhedron, and the interior of any one triangle is one of the *faces* of the polyhedron.

We shall refer to the polyhedron as the polyhedron p or $[t]$ as may be convenient.

It follows at once from the definition that no two triangles of a polyhedron lie in the same plane and have an interior point in common; for if this were the case, each triangle could be so decomposed that two triangles resulting from the decomposition would coincide, which is contrary to (2) of the definition.

For the purposes of this discussion the triangles of the polyhedron are regarded as decomposed so that (1) no two triangles have an interior point in common, (2) no edge of one triangle lies within another triangle. Such decompositions are always possible; for (1) if two triangles have an interior point in common, they have a segment in common, and this segment, extended if necessary, decomposes each of the triangles into two polygons, and these may in turn be decomposed into triangles. (2) If a side of one triangle lies within another, this side, extended if necessary, may be considered a decomposing segment.

The process of decomposing the triangles of the polyhedron in this manner may be called the normalizing process. The finitude of the process follows at once from the finitude of the number of triangles in the set we are considering.

36. *A plane separates space into two connected entirely open sets.*

37. *If two planes have one point in common, they have a second point in common and hence a line in common.*

38. *A plane has in common with a polyhedron at most a finite set of triangles with their interior points, and a finite set of segments, together with a finite set of points.*

That these include all of the polyhedron which can lie in the plane follows directly from the definition. Obviously any or all of these sets may be nonexistent.

DEFINITIONS. That part of a plane which lies on one side of a line in it is called a *half-plane* and is said to proceed from this line.

Two non-coincident half-planes proceeding from the same line and not lying in the same plane form a *dihedral angle.* The two half-planes are called the *faces* of the angle, and the line from which they proceed, its *edge.* If the two faces are α and β, the angle is denoted by (α, β).

39. *A dihedral angle separates three-space into two connected entirely open sets.*

One is called the interior and the other the exterior set.

40. *If A is a point in a face α of a dihedral angle (α, β), and if B is on the same side of the plane determined by α as the face β, then the segment AB lies wholly or partly within (α, β).*

§ 6. *The Separation of Three-Space by a Polyhedron.*

41. *If $[t]'$ and $[t]''$ are complementary subsets of the set $[t]$, then there is at least one edge which is a side of an odd number of triangles in each set.*

PROOF. If every side of a triangle in $[t]'$ which is also a side of a triangle $[t]''$ is a side of an even number of triangles in $[t]'$, then every side of a triangle in $[t]'$ is a side of an even number of triangles in it, which is contrary to (2) of the definition.

42. *For any line l and any finite set of points $[P]$ not on l there is a plane through l not containing a point of $[P]$.*

43. *If a plane contains no three non-collinear vertices of a polyhedron, the plane has in common with it a finite set of segments with their end-points, together with a finite set of points.*

The finite set of points and also the set of segments may of course be non-existent.

44. *A plane which contains no vertex of a polyhedron has a finite set of simple polygons in common with it.*

PROOF. If the plane contains no point of the polyhedron, the theorem is verified, the number of polygons being zero.

If the plane contains points of the polyhedron, it must meet an edge, and the interior of each triangle of which this edge is a side has a segment in common with the plane (37). Hence the plane and the polyhedron have in common a set of segments such that every end-point is the end-point of an even number of segments. Starting with any one of these segments, we can trace a broken line until we return to some point in the line already traced. Since every vertex is the end-point of an even number of segments, this process may be repeated until the whole set of segments is exhausted, thus tracing a finite set of polygons.

45. *An angle not containing a vertex or a point in an edge of a polyhedron contains an even number of points in it, provided the vertex of the angle does not lie on the polyhedron.*

PROOF. (a) Suppose no vertex of the polyhedron lies in the plane of the angle. Then by (44) the polyhedron and the plane have a finite set of simple polygons in common; and since the angle does not contain a vertex of one of them (the angle contains no point of an edge of the polyhedron), then by (18) the angle contains an even number of points in each polygon and hence in the set of all of them.

(b) If the plane of the angle contains a vertex of the polyhedron, then by (42) obtain a point P such that the planes determined by the point P and each side of the given angle contain no such vertex. Let the given angle be (h_1, h_2) and let k be to the ray proceeding from the given point through P. Then we have two angles, (h_1, k) and (h_2, k), such that each contains an even number of points of the polyhedron. Hence if there is an even number of points on k, there is an even number of points on both h_1 and h_2, and hence on the angle (h_1, h_2). If there is an odd number of points of the polyhedron on k, there is an odd number of such points on h_1 and also h_2, and hence an even number on the angle (h_1, h_2).

DEFINITION. A point not on a polyhedron is *interior* or *exterior* to it according as a half-line proceeding from it and not meeting an edge or a vertex meets it in an odd or even number of points. Two points which are both exterior or both interior are said to lie on the same side of the polyhedron.

It follows from (45) that the interior and exterior quality of a point as here defined depends upon the point and the polyhedron and not upon the particular ray chosen, and also that every point not on the polyhedron is either interior or exterior.

46. *Any two points connected by a broken line which does not meet a polyhedron are both exterior or both interior to it.*

PROOF. Let AB be any segment of the broken line. If at least one of the rays AB and BA contains no vertex or point in an edge of the polyhedron, then by the definition the points of the segment AB, including its end-points, are all exterior or all interior.

If both rays AB and BA contain a vertex or a point of an edge, proceed as follows:

Let α be a plane containing the line AB but containing no vertex of the polygon except such as may lie on the line AB. Since the plane α contains no three non-collinear vertices of the polyhedron, the plane and the polyhedron have only a finite set of segments in common (43), and hence by (15) we can find a point D such that no point of the polyhedron lies within the triangle ABD. Within this triangle select any point C such that neither of the rays AC and BC contains a vertex or a point of an edge of the polyhedron. Then by the definition A and C are on the same side of the polyhedron, as are also B and C. Hence A and B are on the same side of it.

Proceeding in this manner, we can then show that the end-points of the broken line are on the same side of the polyhedron.

It is an immediate consequence of the preceding that:

47. *The points of a broken line which does not meet a polyhedron are all interior or all exterior points of the polyhedron.*

And also:

48. *A broken line connecting an interior and an exterior point meets the polyhedron.*

DEFINITION. Any two points connectible by a broken line exterior to the polyhedron have *external accessibility,* and if connectible by an interior broken line they have *internal accessibility.*

49. *Two points in the same face of a polyhedron or on the triangle enclosing this face have both internal and external accessibility.*

PROOF. Let A and B be the given points lying in a face t_1 of a polyhedron p. Denote by α the plane determined by t_1. Through A and B pass a plane β containing no vertex of p except such as may lie in the line AB. In β draw a line l through A meeting no edge or vertex of p. This is possible, since no edge lies in β except possibly in the line AB. Then by (43), (15) and (11) there is a point C on l such that no point of the segments AC and BC lies on p, and further such that the ray BC meets no edge or vertex of p. Then AC and BC and the point C are entirely interior or entirely exterior.

Similarly there is a point D on l in the order CAD such that AD and DB and D are entirely interior or entirely exterior. But if C is exterior, D is interior by the definition of interior and exterior. Hence A and B have both internal and external accessibility. If A is a vertex or lies in an edge, it may be connected with a point P in the face by passing a plane γ through AP distinct from α, then ordering the segments in γ consisting of edges of the polyhedron or intersections with its faces, and applying (15) and (11).

50. *If B is a point of a face of a polyhedron and AB and CB are segments both entirely interior or both entirely exterior, then A and C are mutually accessible with respect to the polyhedron.*

PROOF. Let t_1 be the face of the polyhedron in which B lies. Then the segments AB and CB are readily shown to lie on the same side of the plane determined by t_1. In the plane of AB and CB apply successively (38), (15), (11) and (12), from which the theorem follows.

51. *If a segment AB meets a face of a polyhedron in a point C and meets the polyhedron in no other point, then AC is entirely interior and CB entirely exterior, or AC entirely exterior and BC entirely interior.*

PROOF. This is an immediate consequence of (50).

52. *If in a polyhedron all faces having a common side are divided into two sets each containing an odd number of faces, then there are two points, one in a face of each set, which are internally accessible, and two points, one in each set, which are externally accessible.*

PROOF. Let AB be the common side and Q a point in it. Through Q pass a plane α not containing a vertex of the polyhedron p. Then the faces whose common side is AB will intersect α in a set of segments radiating from Q. Order these segments as in (10) and denote them by σ_i. In the plane α contruct a triangle t with Q as an interior point which does not meet p except in points on the segments σ_i (17). The triangle t is also to be such that the lines determined by its sides meet no vertex or edge of p and no vertex of the triangle lies on a segment σ_i.

Let the intersection points of t and σ_i be A_i. Suppose the segment (or broken line) A_1A_2 is exterior; then A_2A_3 is interior by the definition of interior and exterior. Similarly A_3A_4 is exterior, etc.

If now the faces of which AB is a common edge are divided into two sets, the segments σ_i are divided into corresponding sets. Let one of these sets be σ_i'. Then at some point in the ordered sequence of segments σ_i an odd number of consecutive segments belong to σ_i', for otherwise σ_i' could not consist of an odd number of segments. Let the segment (or broken line) $A_{h-1}A_h$ which connects the first of these with a segment not of σ_i' be exterior; then, since these are alternately interior and exterior, it follows that the segment A_kA_{k+1}, which connects the last of these segments with a segment not of $[\sigma]'$, is interior; and hence the theorem is proved.

53. *Any two points of a polyhedron have both internal and external mutual accessibility.*

PROOF. Let A be any point on the polyhedron. It follows from (49) and (50) that if a single point of a face of the polyhedron is accessible from A, then every point of this face is thus accessible. Suppose not all points of the polyhedron accessible from A. Let $[t]'$ be the set whose faces are thus accessible, and $[t]''$ its complementary subset of $[t]$. Then by (41) there is at least one edge which is a side of an odd number of triangles in each of these sets. Then by (52) points of some face of $[t]''$ are accessible with points of some face of $[t]'$ by both interior and exterior connection. Hence it follows that every point of the polyhedron is accessible from A by both interior and exterior connection.

54. *Any two interior points and also any two exterior points of a polyhedron are mutually accessible with respect to the polyhedron.*

PROOF. Let A and B be any two exterior points of a polyhedron p. Connect them to points in faces of the polyhedron by segments AC and BD such that the rays AC and BD do not contain a vertex or a point of an edge of p, and the segments AC and BD contain no points of p. By (53) C and D are mutually accessible by exterior connection, and hence by (50) A and B are thus accessible.

If the given points are both interior, the argument is precisely the same.

From (48), (54), and from the fact that every point not on a polyhedron is either interior or exterior to it, we now have:

55. *A polyhedron separates the remaining points of a three-space into two connected sets.*

It is also easy to show that these sets are both entirely open, the definition of entirely open set being modified to suit the case of three-dimensional space.

56. *For every polyhedron there is a complete line which is entirely exterior to it, while no line is entirely interior to it.*

PROOF. This follows at once from (29) by passing a plane through the polyhedron.

57. *If an interior point of a polyhedron lies within a dihedral angle, then a point of the polyhedron lies within the dihedral angle.*

PROOF. Let l be the edge of the dihedral angle and P the given point. Connect P with a point A on l exterior to the polyhedron (56). Then the segment AP lies entirely within the dihedral angle and meets the polyhedron (54).

§ 7. *Decomposition of the Polyhedron.*

DEFINITION. A finite set $[p]$ of polyhedrons is said to constitute a *decomposition* of a polyhedron p if:

(a) No two polyhedrons of $[p]$ have an interior point in common.

(b) Every interior point of a polyhedron of $[p]$ is an interior point of p.

(c) Every interior point of p is an interior point of a polyhedron of $[p]$ or lies on one of these polyhedrons.

58. *If $[p]$ is a decomposition of a polyhedron p, and if P is a point on a polyhedron p_1 of $[p]$, then P is an interior point of p or lies on this polyhedron.*

PROOF. Let A be any interior point of p_1. Connect A and P by a broken line which lies entirely within p_1 (55). Then by the definition of decomposition the broken line AP lies within p and hence contains no point of p. Therefore P can not be an exterior point of p, since every broken line connecting an interior and exterior point meets the polygon (48).

59. *Not every polyhedron can be decomposed into tetrahedrons in such manner that every vertex of the tetrahedrons is a vertex of the polyhedron.*

PROOF. This proposition is proved by exhibiting a polyhedron of which no such decomposition exists.

Let E_1, C, E_2 (in Fig. 1) be any three non-collinear points. S_1 and S_2 are points on CE_1 and CE_2 respectively in the orders CE_1S_1 and CE_2S_2. Let O be the intersection point of the segments S_1E_2 and S_2E_1. (That these segments meet is an immediate consequence of the triangle transversal axiom.) V_1 and V_2 are points on the segments OS_1 and OS_2 respectively. Connect S_1S_2. Let the intersections of S_1S_2 with the lines CV_1 and CV_2 be O_1 and O_2 respectively. It follows that we have the orders CV_1O_1 and CV_2O_2. A is any point not in the plane E_1CE_2. Connect O, O_1 and O_2 with A. On segments AO_1 and AO_2 select points D_1 and D_2 respectively, and on the line AO select B in the order AOB. Then we consider the following points, triangles and segments:

$$\text{Seven points:}\quad \begin{cases} \qquad\qquad D_1,\ E_1, \\ A,\ B,\ C,\ D_2,\ E_2. \end{cases}$$

$$\text{Ten triangles:}\quad \begin{cases} ABD_1,\ ACD_1,\ CD_1E_1,\ BD_1E_1,\ BE_1E_2, \\ ABD_2,\ ACD_2,\ CD_2E_2,\ BD_2E_2,\ CE_1E_2. \end{cases}$$

$$\text{Fifteen segments:}\quad \begin{cases} AB,\ BD_1,\ CD_1, \\ AC,\ BD_2,\ CD_2,\ D_1E_1, \\ AD_1,\ BE_1,\ CE_1,\ D_2E_2, \\ AD_2,\ BE_2,\ CE_2. \end{cases} E_1E_2,$$

These triangles form a polyhedron; for:

(1) Every side of a triangle is common to exactly two triangles.

(2) No subset of these triangles exists such that (1) of the definition of a polyhedron holds. This remains true for any finite decomposition of the triangles here given, for every such decomposition leaves every edge common to exactly two triangles.

We now show that any triangle not a triangle of this polyhedron but determined by three of these points lies wholly or partly outside the polyhedron. This is best done by showing that every segment not an edge of this polyhedron but determined by two of its vertices lies wholly or partly outside the polyhedron. These segments are:

$$\begin{cases} AE_1,\ D_1E_2, \\ AE_2,\ D_2E_1. \end{cases} BC,\ D_1D_2,$$

All vertices of the polyhedron except A, C, E_1 lie on the same side of the plane determined by these points. Hence there is no interior point of the polyhedron in this plane, and the only points of the polyhedron in this plane are A, C, E_1 and those of the segments AC and CE_1. Hence the segment AE_1 is entirely exterior to the polyhedron. In the same manner AE_2, BC and D_1D_2 may be shown to be entirely exterior to the polyhedron.

We now notice that there are no points of the polyhedron within the dihedral angle whose edge is AO and whose faces contain the points O_1 and O_2. Hence by (57) there is no interior point of the polyhedron within this dihedral angle. Further, E_1 and D_1 lie on the same side of the plane AOD_2. Hence by (40) part of the segment D_2E_1 lies partly with the dihedral angle O_1AOO_2. But this part of D_2E_1 is exterior to the polyhedron. In the same manner we may show that D_1E_2 is partly exterior to the polyhedron.

We now observe that every triangle determined by three of the points A, B, C, D_1, D_2, E_1, E_2 which is not a triangle of the given polyhedron has one of these segments as a side. Hence by (58) no such triangle can belong to a set of triangles which constitute a set of polyhedrons that form a decomposition of the given polyhedron.

DEFINITION. A polyhedron is *convex* if all its interior points lie on the same side of every plane determined by its faces.

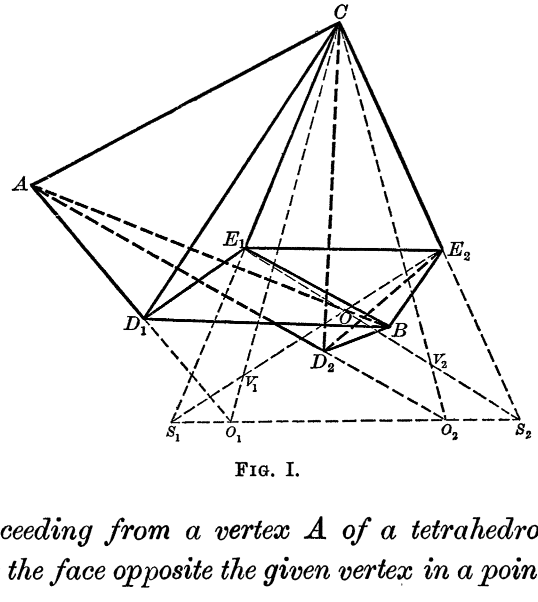

FIG. I.

60. *A ray proceeding from a vertex A of a tetrahedron and containing an interior point P meets the face opposite the given vertex in a point B in the order APB.*

61. PROBLEM. *To decompose any convex polyhedron into a set of tetrahedrons.*

SOLUTION. Let A be any vertex of a given convex polyhedron p. Consider all triangles of p which do not lie in the same plane with a triangle of p of which A is a vertex. Any one of these triangles, together with the three triangles determined by its sides and the point A, form a tetrahedron. The set $[\tau]$ of all such tetrahedrons form a decomposition of p; for:

(a) *No two of these tetrahedrons have an interior point in common.*

Suppose there is such common interior point P. Then the ray AP must meet p in two different faces (60). Since no two faces have a point in common, it would follow by (53) that not all interior points of the polyhedron lie on the same side of each of the planes determined by these faces.

(b) *Every interior point of p lies on or within a tetrahedron of* $[\tau]$.

Let P be any interior point of p. Then the ray AP meets p in a point not in a plane with triangles of which A is a vertex, and hence it meets a face, an edge or vertex of a tetrahedron of $[\tau]$. That is, P is on or within one of these tetrahedrons.

(c) *Every point within a tetrahedron of* $[\tau]$ *lies within* p.

Let P be any point within a tetrahedron of $[\tau]$. Then the ray AP meets a face of p in a point B in the order APB (60), and in no other point. Hence, by the definition of interior and exterior points P is an interior point of p.

62. *If a plane α contains an interior point of a polyhedron p, then there exists a set of triangles,* $[t]_{p,\alpha}$, *such that every interior point of p which lies in α is on or within a triangle of* $[t]_{p,\alpha}$, *while no exterior point lies within one of these triangles.*

PROOF. If α contains a face of p, then an interior point of p can be connected with a point in this face only by crossing a vertex or edge of p (9), provided the connecting broken line lies in α. Denote by $[\sigma]$ the set of segments, and their end-points, consisting of all edges of p in α and all intersections of α with faces of p.

Let P be any interior point of p in α and $[P]$ all interior points of p in α accessible from P with respect to $[\sigma]$. By (56) this set does not include the whole plane. Hence there is a subset of $[\sigma]$ which fulfils the conditions of (27), and hence forms a simple polygon p_1. All points of this polygon are points of p, and all its interior points are interior points of p.

If there is an interior point P_1 of p in α not within p_1, then in the same manner as above we obtain another polygon p_2 having similar properties, and so on. These polygons can not have an interior point in common, and hence a segment can be a side common to not more than two of them. Hence the process exhausts the segments of $[\sigma]$ and we obtain a finite set of polygons such that every interior point of p in α lies within one of them. A decomposition of these polygons into triangles gives the set of triangles specified in the theorem.

63. *If a plane α contains an interior point of a polyhedron p, then* $[t]_{p,\alpha}$, *together with* $[t]$ *of p as decomposed by α, form a decomposition of p.*

PROOF. Consider those triangles of p which lie on one side of the plane α together with $[t]_{p,\alpha}$ and their interior points. Denote this set by $[t]'$. If a side of a triangle in this set does not lie in α, then by the definition of p every such segment is a side of an even number of triangles.

If a side of a triangle is an intersection of α with a face of p, then it is common to exactly two triangles, one of p and one of $[t]_{p,\alpha}$; and if the side lies in α but not in p, it is also common to two triangles.

It remains to consider the case when an edge of p lies in α. Denote the segment by σ_1. We need to consider two cases: when σ_1 is a side of a triangle in $[t]_{p,\alpha}$, and when it is not. In either case it follows directly from the construction used in the proof of (52) that σ_1 is a side of an even number of triangles in $[t]'$.

Obviously no two triangles of $[t]'$ have an interior point in common, and no one contains an interior point of another. Since the set $[t]'$ is finite, it follows that there must be some subset of it of which no proper subset satisfies the condition that every side of a triangle is common to an even number of triangles. Hence the set $[t]'$ constitutes a set of polyhedrons $[p]'$ such that no two of them have a face in common. Then:

(a) No two polyhedrons of $[p]'$ have an interior point in common. To show this, note first that every polyhedron of $[p]'$ has at least one face in α, and that each triangle of $[t]_{p,}$ is used only once in $[p]'$. From a point Q within a triangle of $[t]_{p,\alpha}$ pass a half-line which contains no edge or vertex of p and which lies on the same side of α as the interior points of $[p]'$. Then there is a point R on this half-line such that there is no point of p on QR. Then QR lies within exactly one polyhedron of $[p]$. It now follows at once that no point on this ray lies within more than one polyhedron of $[p]'$.

(b) It follows at once also that every interior point of p on the same side of α as R lies within a polyhedron of $[p]'$, and

(c) That every interior point of a polyhedron of $[p]'$ is an interior point of p.

If we now consider that part of p which lies on the opposite side of α from R, we obtain a similar set of $[p]''$ of polyhedrons. Now all interior points of p not in α lie within polyhedrons of the sets $[p]'$ and $[p]''$, and all interior points of p not within these lie on or within the set of triangles $[t]_{p,\alpha}$; that is, they are on the polyhedrons of $[p]'$ and $[p]''$. Hence these two sets form a decomposition of p.

64. PROBLEM. *To decompose any polyhedron into a set of tetrahedrons.*

SOLUTION. If the polyhedron is convex, the problem is solved in (61). If the polyhedron is not convex, the solution is made by indicating how to decompose it into a set of convex polyhedrons.

Let α be a plane determined by a face of the polyhedron p such that not all of the interior points of p lie on the same side of α. Then by (63) $[t]_{p,\alpha}$ decompose p into a set of polyhedrons such that all interior points of each polyhedron lie on the same side of α. If some of the resulting polyhedrons are not convex, we decompose each as before, etc. That this will result in a set of convex polyhedrons follows from the finitude of the number of edges of p.

§ 8. *Remarks on the Definition of the Polyhedron.*

DEFINITION. A polyhedron consists of a finite set $[t]$ of triangles, together with their interior points, such that

(1) Every side of a triangle is common to an even number of triangles.

(2) There exists no finite decomposition of the triangles of $[t]$ into other triangles forming a set $[t]'$ such that (1) is true of a proper subset of $[t]'$.

65. *This definition is not redundant.*

(1) is independent of (2), because (2) is satisfied by a single triangle while (1) is not.

(2) is independent of (1), because (1) is satisfied by the set of triangles forming two distinct tetrahedrons, while (2) is not.

66. (1) *of the definition is not provable from* (2) *and the following statement: "Every side of a triangle is common to at least two triangles."*

PROOF. (2) of the definition and the statement "Every side of a triangle is common to at least two triangles" are both satisfied by the following figure, while (1) is not.

Consider an anchor ring with the opening through it closed by that part of a plane which is bounded by the smallest circle whose plane divides the ring into two complete rings. Connect the boundary of a small circle within this circle with the body of the anchor ring by means of a tube which shall contain no interior points of the large circle except the small circle in its interior. If these surfaces are formed by triangles instead of by continuously curved surfaces, we have a set of triangles which satisfy every condition of the definition except that its edges are not all common to an even number of triangles. Obviously this set of triangles does not separate space into two sets.

In terms of the language of this paper a figure corresponding to the above may be described as follows:

Consider a tetrahedron $OABC$. Through A_1, B_1, C_1, points on AO, BO, CO respectively, pass a plane. Let the three non-collinear points A_2, B_2, C_2 within the triangle ABC and the point O determine three triangles. Let the segments OA_2, OB_2, OC_2 meet the interior of the triangle $A_1B_1C_1$ in the points A_3, B_3 and C_3 respectively. Arrange the notation so that the segment A_1A_3 does not meet A_3B_3, B_3C_3 or C_3A_3, and so that AA_2 does not intersect A_2B_2, B_2C_2 or C_2A_2. A polyhedron is formed by the triangles into which the following polygons may be decomposed: $ACBB_2C_2A_2A$, AA_2B_2BA, $A_1C_1B_1B_3C_3A_3A_1$, $A_1A_3B_3B_1A_1$, ABB_1A_1A, BCC_1B_1B, CAA_1C_1C, $A_2B_2B_3A_3A_2$, $B_2C_2C_3B_3B_2$ and $C_2A_2A_3C_3C_2$. (This polyhedron corresponds to the anchor ring.)

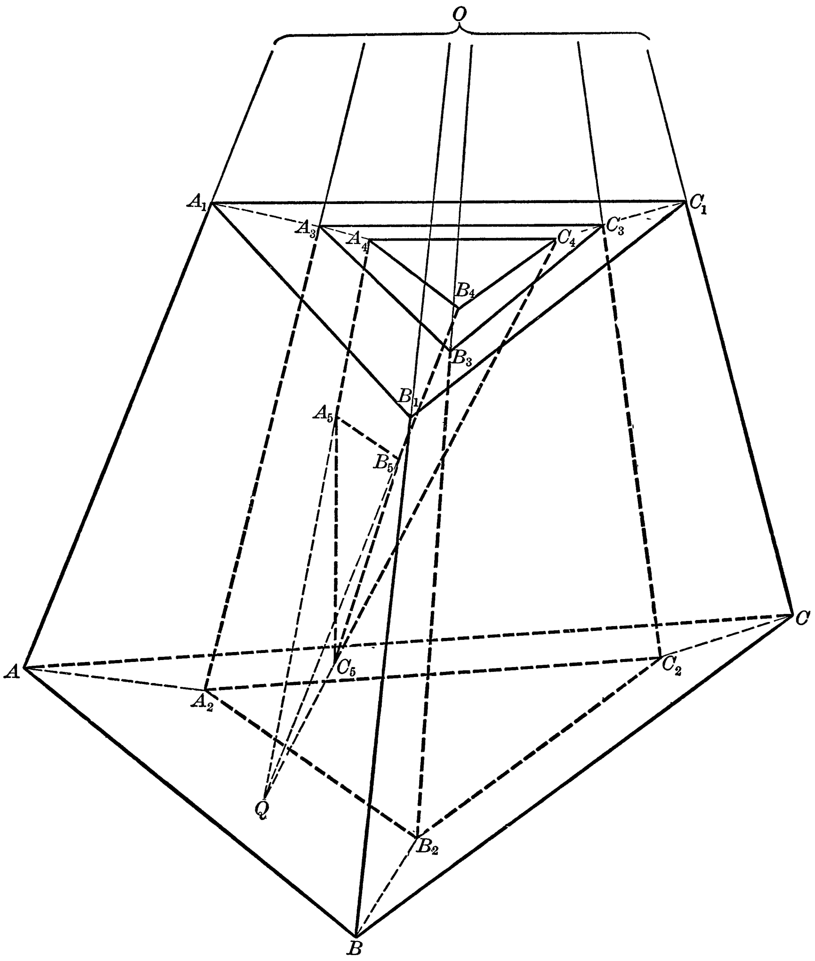

FIG. 11.

Call this set of triangles $[t]$. From Q, any interior point of AA_2B_2BA, draw segments to the non-collinear points A_4, B_4, C_4 within the triangle $A_3B_3C_3$, meeting the interior of the quadrilateral $A_2B_2B_3A_3A_2$ in the points A_5, B_5, C_5 respectively. We now adjoin to the set of triangles $[t]$, suitably decomposed, the set of triangles into which we may decompose $A_4B_4B_5A_5A_4$, $B_4C_4C_5B_5B_4$ and $C_4A_4A_5C_5C_4$, and the remainder of the interior of $A_3B_3C_3$ when $A_4B_4C_4$ is removed, and omit the triangle $A_5B_5C_5$. This set of triangles have the following properties:

(1) Every side of a triangle is common to two or more triangles. (All are common to two triangles except A_3B_3, B_3C_3, C_3A_3, each of which is common to three triangles.)

(2) There is no proper subset of which (1) is true.

BOSTON, MASS., *January* 11, 1910.